BINGO - Supplementary Material

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Appendix A

In the paper we have found the following expression for $EU_a^{N-1}(\omega, r_1, b_2)$ in a split-the-pie domain (Eq. (12) in the paper):

$$EU_a^{N-1}(\omega, r_1, b_2) \quad = \quad \begin{cases} \frac{1}{b_2} \cdot (1 - u_1(\omega)) \cdot (u_1(\omega) - r_1) + r_1 & \text{if } 1 - b_2 \leq u_1(\omega) \\ u_1(\omega) & \text{otherwise} \end{cases}$$

Our goal is to find the offer ω that maximizes this value.

To simplify notation, we will now write $u_1(\omega)$ as x and $EU_a^{N-1}(\omega, r_1, b_2)$ as $f(x)$. So, the goal is to find the value of x that maximizes the following expression on the interval $[0, 1]$.

$$f(x) = \begin{cases} x & \text{if } x \leq 1 - b_2 \\ \frac{1}{b_2} \cdot (1 - x) \cdot (x - r_1) + r_1 & \text{if } x \geq 1 - b_2 \end{cases} \tag{16}$$

To find the maximum of this function, we first have to find its maxima on the two respective intervals $[0, 1 - b_2]$ and $[1 - b_2, 1]$ separately. For the first interval it is clear that the maximum is attained at $x = 1 - b_2$, and its value there is $f(x) = 1 - b_2$.

For the second interval, we first rewrite its expression as:

$$\begin{aligned} f(x) \quad &= \quad \frac{1}{b_2} \cdot (1 - x) \cdot (x - r_1) + r_1 \\ &= \quad \frac{1}{b_2}(-x^2 + x + xr_1 - r_1) + r_1 \end{aligned}$$

Then, to find its maximum (which we will denote by $\hat{x}$) we calculate its derivative:

$$\frac{\partial f}{\partial x} = \frac{-2x + (1 + r_1)}{b_2}$$

Setting this derivative to zero we find: $-2x + (1 + r_1) = 0$, from which we get: $\hat{x} = \frac{1}{2} + \frac{1}{2}r_1$. Note that the second derivative is $-\frac{2}{b_2}$, which is strictly negative, which means that $\hat{x}$ is indeed a *maximum* (as opposed to a minimum).

Now, the maximum $\hat{x}$ of the quadratic expression is not necessarily the maximum of the function f (which we will denote by x_{max}), because $\hat{x}$ might fall outside the domain $[1 - b_2, 1]$ where the quadratic expression is valid. So, we need to check whether or not $\hat{x} \in [1 - b_2, 1]$. To do this, first note that since $r_1 < 1$, we immediately have that $\frac{1}{2} + \frac{1}{2}r_1 < 1$, so indeed $\hat{x} < 1$. So, we now only need to check that $\hat{x} \geq 1 - b_2$. It is easy to see that this is true if and only if $r_1 \geq 1 - 2b_2$:

$$\begin{aligned} \frac{1}{2} + \frac{1}{2}r_1 \quad &\geq \quad 1 - b_2 \\ \frac{1}{2}r_1 \quad &\geq \quad \frac{1}{2} - b_2 \\ r_1 \quad &\geq \quad 1 - 2b_2 \end{aligned}$$

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In other words, we now have that $x_{max} = \hat{x}$ if and only if $r_1 \leq 1 - 2b_2$.

In the case that $r_1 > 1 - 2b_2$, which implies that $\hat{x} < 1 - b_2$, we have that the quadratic expression is strictly decreasing after $x = 1 - b_2$ (see Figure 1, left), which means that the overall maximum of f lies exactly at $x = 1 - b_2$.

Combing these two results we get:

$$x_{max} = \begin{cases} 1 - b_2 & \text{if } r_1 \leq 1 - 2b_2 \\ \frac{1}{2} + \frac{1}{2}r_1 & \text{if } r_1 \geq 1 - 2b_2 \end{cases} \tag{17}$$

Finally, since the condition $r_1 \leq 1 - 2b_2$ is equivalent to $\frac{1}{2} + \frac{1}{2}r_1 \leq 1 - b_2$, we see that Eq. (17) can be rewritten as:

$$x_{max} = \max \left\{ \frac{1}{2} + \frac{1}{2}r_1 \ , \ 1 - b_2 \right\}$$

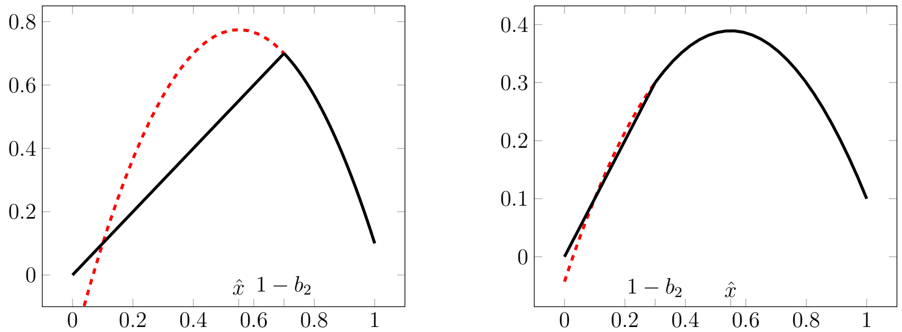

Figure 1. Left: plot of Equation (16) for the values $r_1 = 0.1$ and $b_2 = 0.3$. The solid black line represents the actual function f, while the dashed red line represents the part of the quadratic expression outside the region where it is valid (i.e. for $x < 1 - b_2$). Right: the same, but now for the values $r_1 = 0.1$ and $b_2 = 0.7$.

Appendix B

We here prove Theorem 3 of the paper.

We will use b_2' as a shorthand for b_2^{N-2} and b_2 as a shorthand for b_2^{N-1}.

To prove the theorem, we will first calculate $U_p^{N-1}(r_1, r_2, b_2)$ (i.e. the anticipated utility for α_2 when α_1 makes the optimal proposal in round $N - 1$). We start from Eq. (9) of the paper, and use the fact that in a split-the-pie-domain we have $u_2(\omega) = 1 - u_1(\omega)$, for any offer ω. So we get:

$$U_p^{N-1}(r_1, r_2, b_2) = \begin{cases} 1 - u_1(\omega^{*N-1}) & \text{if } r_2 < 1 - u_1(\omega^{*N-1}) \\ r_2 & \text{otherwise} \end{cases}$$

Combining this with Eq. (8) from the paper and Eq. (17) we get:

$$U_p^{N-1}(r_1, r_2, b_2) = \begin{cases} b_2 & \text{if } r_2 < b_2 \text{ and } r_1 \leq 1 - 2b_2 \\ \frac{1}{2} - \frac{1}{2}r_1 & \text{if } r_2 < \frac{1}{2} - \frac{1}{2}r_1 \text{ and } r_1 \geq 1 - 2b_2 \\ r_2 & \text{otherwise} \end{cases}$$

However, the inequality $r_2 < b_2$ always holds, by definition of b_2 (see Eq. (3) in the paper) and by the assumption that agents are rational, so it can be removed. Furthermore, note that the condition $r_2 < \frac{1}{2} - \frac{1}{2}r_1$ can be rewritten as $r_1 < 1 - 2r_2$, so we get:

$$U_p^{N-1}(r_1, r_2, b_2) = \begin{cases} b_2 & \text{if } r_1 \leq 1 - 2b_2 \\ \frac{1}{2} - \frac{1}{2}r_1 & \text{if } r_1 \in [1 - 2b_2, 1 - 2r_2] \\ r_2 & \text{if } r_1 \geq 1 - 2r_2 \end{cases} \tag{18}$$

Now, to continue with our proof of Theorem 3, recall that ω^{*N-1} is the offer that maximizes EU_a^{N-1}, which means that by definition it satisfies $u_1(\omega^{*N-1}) = x_{max}$ (see Appendix A). So, the expression for $EU_a^{N-1}(\omega^{*N-1}, r_1, b_2)$ can be found by replacing $u_1(\omega)$ in Eq. (12) by the expression for x_{max} as given by Eq. (17). This yields:

$$EU_a^{N-1}(\omega^{*N-1}, r_1, b_2) = \begin{cases} 1 - b_2 & \text{if } r_1 \leq 1 - 2b_2 \\ \frac{1}{4b_2} \cdot \Big(1 - r_1\Big)^2 + r_1 & \text{if } r_1 \geq 1 - 2b_2 \end{cases}$$

Combining this with Eqs. (10) and (11) from the paper, we get:

$$U_a^{N-2}(\omega, r_1, r_2, b_2') \quad = \quad \begin{cases} u_2(\omega) & \text{if } r_1 \leq 1-2b_2 \text{ and } 1-b_2 \leq 1-u_2(\omega) \\ u_2(\omega) & \text{if } r_1 \geq 1-2b_2 \text{ and } \frac{1}{4b_2}\cdot\Big(1-r_1\Big)^2 + r_1 \leq 1-u_2(\omega) \\ U_p^{N-1}(r_1, r_2, b_2) & \text{otherwise} \end{cases}$$

Note that if α_2 proposes ω in round $N-2$ then we must have $b_2 \leq u_2(\omega)$, by Eq. (3) from the paper. This means that the condition $1-b_2 \leq 1-u_2(\omega)$ can only hold if $b_2 = u_2(\omega)$.

Furthermore, for the same reason, we note that the condition $\frac{1}{4b_2}\cdot\Big(1-r_1\Big)^2 + r_1 \;\leq\; 1-u_2(\omega)$ implies:

$$\frac{1}{4b_2}\cdot\Big(1-r_1\Big)^2 + r_1 \;\leq\; 1-b_2 \tag{19}$$

This can be rewritten as:

$$\begin{aligned} \frac{1}{4b_2}\cdot\Big(1-r_1\Big)^2 + (r_1-1) + b_2 &\leq 0 \\ (1-r_1)^2 + 4b_2\cdot(r_1-1) + 4b_2^2 &\leq 0 \\ ((r_1-1)+2b_2)^2 &\leq 0 \end{aligned}$$

which clearly can only hold if $(r_1-1)+2b_2 = 0$, which means we have:

$$r_1 = 1-2b_2$$

Furthermore, we again must have $b_2 = u_2(\omega)$ because otherwise the inequality in Eq. (19) would be strict and then there would be no solution at all. We are therefore left with:

$$U_a^{N-2}(\omega, r_1, r_2, b_2') \quad = \quad \begin{cases} u_2(\omega) & \text{if } r_1 \leq 1-2b_2 \text{ and } b_2 = u_2(\omega) \\ u_2(\omega) & \text{if } r_1 = 1-2b_2 \text{ and } b_2 = u_2(\omega) \\ U_p^{N-1}(r_1, r_2, b_2) & \text{otherwise} \end{cases}$$

but now we see that the second condition is just a special case of the first one, so it can be omitted:

$$U_a^{N-2}(\omega, r_1, r_2, b_2') \quad = \quad \begin{cases} u_2(\omega) & \text{if } r_1 \leq 1-2b_2 \text{ and } b_2 = u_2(\omega) \\ U_p^{N-1}(r_1, r_2, b_2) & \text{otherwise} \end{cases}$$

Finally, when we look at Eq. (18) then we see that if the first condition holds, then $u_2(\omega)$ actually equals $U_p^{N-1}(r_1, r_2, b_2)$, so we can just as well write:

$$U_a^{N-2}(\omega, r_1, r_2, b_2') \quad = \quad U_p^{N-1}(r_1, r_2, b_2) \tag{20}$$

Note that it *looks* as if $U_a^{N-2}(\omega, r_1, r_2, b_2')$ does not depend on the offer ω because ω does not appear on the right-hand side. However, this is misleading, because the right-hand side does depend on b_2, which depends on the offer that α_2 proposes.

The next step to determine the optimal offer, is to integrate U_a^{N-2} over all possible values of r_1:

$$\begin{aligned} EU_a^{N-2}(\omega, r_2, b_1', b_2') &= \frac{1}{b_1'}\int_0^{b_1'} U_a^{N-2}(\omega, r_1, r_2, b_2')\, dr_1 \\ &= \frac{1}{b_1'}\int_0^{b_1'} U_p^{N-1}(r_1, r_2, b_2)\, dr_1 \end{aligned}$$

where $b_2 = \min\{u_2(\omega), b_2'\}$.

We see from Eq. (18) that the expression for U_p^{N-1} is different on three separate intervals, which are demarcated by the values $1-2b_2$ and $1-2r_2$. Therefore, to calculate the integral we have to distinguish between six different cases depending on which of these three intervals have any overlap with the interval $[0, b_1']$ over which we are integrating.

Case 1: $0 \;\leq\; b_1' \;\leq\; 1-2b_2 \;\leq\; 1-2r_2$

In this case the interval of integration $[0, b_1']$ is entirely contained within the first interval of Eq. (18), so we get:

$$EU_a^{N-2}(\omega, r_2, b_1', b_2') \quad = \quad \frac{1}{b_1'}\cdot\int_0^{b_1'} b_2\, dr_1 \quad = \quad b_2$$

Case 2: $\quad 1-2b_2 \quad \leq \quad 0 \quad \leq \quad b'_1 \quad \leq \quad 1-2r_2$

In this case the interval of integration $[0, b'_1]$ is entirely contained within the middle interval of Eq. (18), so we get:

$$\begin{aligned} EU_a^{N-2}(\omega, r_2, b'_1, b'_2) &= \frac{1}{b'_1} \cdot \int_0^{b'_1} \frac{1}{2} - \frac{1}{2} r_1 \, dr_1 \\ &= \frac{1}{b_1} \cdot \Big(\frac{1}{2} b'_1 - \frac{1}{4} b'^2_1 \Big) \\ &= \frac{1}{2} - \frac{1}{4} b'_1 \end{aligned}$$

Case 3: $\quad 1-2b_2 \quad \leq \quad 1-2r_2 \quad \leq \quad 0 \quad \leq \quad b'_1$

In this case the interval of integration $[0, b'_1]$ is entirely contained within the last interval of Eq. (18), so we get:

$$EU_a^{N-2}(\omega, r_2, b'_1, b'_2) \quad = \quad \frac{1}{b'_1} \cdot \int_0^{b'_1} r_2 \, dr_1 \quad = \quad r_2$$

Case 4: $\quad 0 \quad \leq \quad 1-2b_2 \quad \leq \quad b'_1 \quad \leq \quad 1-2r_2$

In this case the interval of integration $[0, b'_1]$ (partially) overlaps with the first two intervals of Eq. (18), so we get:

$$\begin{aligned} EU_a^{N-2}(\omega, r_2, b'_1, b'_2) &= \frac{1}{b'_1} \cdot \Big(\int_0^{1-2b_2} b_2 \, dr_1 + \int_{1-2b_2}^{b'_1} \frac{1}{2} - \frac{1}{2} r_1 dr_1 \Big) \\ &= \frac{1}{b'_1} \cdot \Big((1-2b_2) \cdot b_2 + \frac{1}{2} r_1 - \frac{1}{4} r_1^2 \Big|_{1-2b_2}^{b'_1} \Big) \\ &= \frac{1}{b'_1} \cdot \Big(b_2 - 2b_2^2 + \frac{1}{2} b'_1 - \frac{1}{4} b'^2_1 - \frac{1}{2}(1-2b_2) + \frac{1}{4}(1-2b_2)^2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(b_2 - 2b_2^2 + \frac{1}{2} b'_1 - \frac{1}{4} b'^2_1 - \frac{1}{2} + b_2 + \frac{1}{4} - b_2 + b_2^2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(b_2 - b_2^2 + \frac{1}{2} b'_1 - \frac{1}{4} b'^2_1 - \frac{1}{4} \Big) \\ &= \frac{1}{b'_1} \cdot \Big(b_2 - b_2^2 - \frac{1}{4} \Big) + \frac{1}{2} - \frac{1}{4} b'_1 \end{aligned}$$

Case 5: $\quad 1-2b_2 \quad \leq \quad 0 \quad \leq \quad 1-2r_2 \quad \leq \quad b'_1$

In this case the interval of integration $[0, b'_1]$ (partially) overlaps with the last two intervals of Eq. (18), so we get:

$$\begin{aligned} EU_a^{N-2}(\omega, r_2, b'_1, b'_2) &= \frac{1}{b'_1} \cdot \Big(\int_0^{1-2r_2} \frac{1}{2} - \frac{1}{2} r_1 dr_1 + \int_{1-2r_2}^{b'_1} r_2 dr_1 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(\frac{1}{2} r_1 - \frac{1}{4} r_1^2 \Big|_0^{1-2r_2} + (b'_1 - (1-2r_2)) \cdot r_2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(\frac{1}{2}(1-2r_2) - \frac{1}{4}(1-2r_2)^2 + (b'_1 - 1) r_2 + 2r_2^2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(\frac{1}{2}(1-2r_2) - \frac{1}{4}(1 - 4r_2 + 4r_2^2) + (b'_1 - 1) r_2 + 2r_2^2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(\frac{1}{2} - r_2 - \frac{1}{4} + r_2 - r_2^2 + (b'_1 - 1) r_2 + 2r_2^2 \Big) \\ &= \frac{1}{b'_1} \cdot \Big(\frac{1}{4} + r_2^2 - (1 - b'_1) r_2 \Big) \end{aligned}$$

Case 6: $\quad 0 \quad \leq \quad 1-2b_2 \quad \leq \quad 1-2r_2 \quad \leq \quad b'_1$

In this case the interval of integration $[0, b'_1]$ has some overlap with all three intervals of Eq. (18), so we get:

$$
\begin{aligned}
EU_a^{N-2}(\omega, r_2, b_1', b_2') &= \frac{1}{b_1'} \cdot \Big(\int_0^{1-2b_2} b_2 \, dr_1 + \int_{1-2b_2}^{1-2r_2} \frac{1}{2} - \frac{1}{2} r_1 dr_1 + \int_{1-2r_2}^{b_1'} r_2 dr_1 \Big) \\
&= \frac{1}{b_1'} \cdot \Big((1-2b_2) \cdot b_2 + \frac{1}{2} r_1 - \frac{1}{4} r_1^2 \Big|_{1-2b_2}^{1-2r_2} + (b_1' - (1-2r_2)) \cdot r_2 \Big) \\
&= \frac{1}{b_1'} \cdot \Big(b_2 - 2b_2^2 + \frac{1}{2}(1-2r_2) - \frac{1}{4}(1-2r_2)^2 - \frac{1}{2}(1-2b_2) + \frac{1}{4}(1-2b_2)^2 + (b_1'-1) r_2 + 2r_2^2 \Big) \\
&= \frac{1}{b_1'} \cdot \Big(b - 2b_2^2 + (b_2 - r_2) - \frac{1}{4}(1 - 4r_2 + 4r_2^2) + \frac{1}{4}(1 - 4b_2 + 4b^2) + (b_1'-1) r_2 + 2r_2^2 \Big) \\
&= \frac{1}{b_1'} \cdot \Big(b_2 - 2b_2^2 + b_2 - r_2 - \frac{1}{4} + r_2 - r_2^2 + \frac{1}{4} - b_2 + b_2^2 + (b_1'-1) r_2 + 2r_2^2 \Big) \\
&= \frac{1}{b_1'} \cdot \Big(b_2 - b_2^2 + r_2^2 - (1-b_1') r_2 \Big)
\end{aligned}
$$

Now, in order to find the optimal offer, we should remark that there is an important difference between the parameters b_1' and r_2 on the one hand, and the parameter b_2 on the other hand. That is, when α_2 chooses her optimal offer, b_1' and r_2 are fixed. On the other hand, the value of b_2 depends on the offer that she chooses.

We therefore group the above six cases together into three larger cases, each corresponding to only the ordering between the fixed values 0, b_1' and $1 - 2r_2$.

- Case A: $1 - 2r_2 \leq 0 \leq b_1'$. This is in fact case 3.
- Case B: $0 \leq 1 - 2r_2 \leq b_1'$. This includes cases 5 and 6.
- Case C: $0 \leq b_1' \leq 1 - 2r_2$. This includes cases 1, 2, and 4.

Case A

Case A happens when $\frac{1}{2} < r_2$. This case is equivalent to Case 3, and we have already seen that in that case we have:

$$
EU_a^{N-2}(\omega, r_2, b_1', b_2') \quad = \quad r_2
$$

This means that it does not matter which offer ω agent α_2 proposes. The reservation value of α_2 is so high that any rational offer is going to be rejected anyway.

Case B

In Case B we have: $0 \leq 1 - 2r_2 \leq b_1'$, which is equivalent to

$$
r_2 \in [\ \frac{1}{2} - \frac{1}{2} b_1' \ , \ \frac{1}{2}]
$$

Combining the two results of Cases 5 and 6 we get:

$$
EU_a^{N-2}(\omega, r_2, b_1', b_2') \quad = \quad \begin{cases} \frac{1}{b_1'} \cdot \Big(b_2 - b_2^2 + r_2^2 - (1-b_1') r_2 \Big) & \text{if } b_2 \leq \frac{1}{2} \\ \frac{1}{b_1'} \cdot \Big(\frac{1}{4} + r_2^2 - (1-b_1') r_2 \Big) & \text{if } \frac{1}{2} \leq b_2 \end{cases} \tag{21}
$$

Recall that the right-hand side depends on ω through:

$$
b_2 \quad = \quad \min \ \{ \ b_2' \ , \ u_2(\omega) \ \}
$$

So, to find the offer ω that maximizes EU_a^{N-2} we need to find the value of b_2 that maximizes it.

So, if we treat this expression as a function of b_2, then we see that it is a concave parabola on the first interval, while it is constant on the second interval. Differentiating the parabola with respect to b_2 and setting its derivative to 0 we get:

$$
1 - 2 \cdot b_2 = 0
$$

which means the maximum of the parabola is attained at $\frac{1}{2}$, and since the function is constant after $b_2 = \frac{1}{2}$, the function is in fact maximal for *any* b_2 with $b_2 \geq \frac{1}{2}$.

In other words: to maximize its expected anticipated utility, α_2 can choose any offer ω with $u_2(\omega) \geq \frac{1}{2}$.

Case C

We have: $0 \leq b_1' \leq 1 - 2r_2$, which is equivalent to stating $r_2 \leq \frac{1}{2} - \frac{1}{2}b_1'$.

Combining the results of cases 1, 2 and 4 we get:

$$EU_a^{N-2}(\omega, r_2, b_1', b_2') \quad = \quad \begin{cases} b_2 & \text{if } b_2 \ \leq \frac{1}{2} - \frac{1}{2}b_1' \\ \frac{1}{b_1'} \cdot \left(b_2 - b_2^2 - \frac{1}{4}\right) + \frac{1}{2} - \frac{1}{4}b_1' & \text{if } \frac{1}{2} - \frac{1}{2}b_1' \leq b_2 \leq \frac{1}{2} \\ \frac{1}{2} - \frac{1}{4}b_1' & \text{if } \frac{1}{2} \leq b_2 \end{cases} \tag{22}$$

Again, we will regard this as a function of b_2. Note that it is linear on the first interval, and constant on the third interval. Furthermore note that its value on the third interval is strictly greater than any value attained on the first interval.

Now, to find the maximum, we need to differentiate the expression for the middle interval. If we do this and then set its derivative to 0 we get, again:

$$1 - 2 \cdot b_2 = 0$$

so the maximum is attained at $x = \frac{1}{2}$. Reasoning in the same way as for Case B we conclude that α_2 can maximize its expected anticipated utility EU_a^{N-2} by proposing any offer ω with $u_2(\omega) \geq \frac{1}{2}$.

Putting the results of Cases A, B, and C together, we have proven the theorem.